XXXII. *Of the Theory of Circulating Decimal Fractions.* *By* John Robertſon, *Lib. R. S.*

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THE great advantages ariſing from the uſe of Arithmetic, particularly in philoſophy and commerce, is ſufficiently known; therefore, every ſtep taken towards its perfection, has always been countenanced by thoſe who were beſt acquainted with its nature and value. On theſe motives I have been induced to offer the annexed paper to this learned Society.

Regiomontanus, it is ſaid, firſt among Europeans, added to the then known arithmetic, an operation by decimal fractions; which he exemplified in his triangular table. Its utility was readily ſeen, and embraced in many nations, and particularly in this; where it appears to have been cultivated in its theory, and facile modes of operation, more than in other places.

Many writers have remarked its excellency in numeral computation, and have pointed out compendiums to avoid the trouble of writing down ſuperfluous figures; particularly in the operations with concrete numbers, or thoſe relative to money, weight, and meaſure; where the gradations from one denomination to another do not proceed in an uniform progreſſion.

In finding the decimal values of the fractional parts of concrete, and other numbers, it often hap-

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pens, that thoſe decimals do not terminate, or end, with a few figures only; and ſometimes are infinite, or never end; and among theſe are many which have one or more figures conſtantly recurring; as in the following proportions, *viz.*

3 : 2 :: 1,0000, &c. : 0,6666, &c.
and 12 : 5 :: 1,0000, &c. : 0,4166, &c.
alſo 7 : 3 :: 1,0000, &c. : 0,428571,428571, &c.

In operations, with ſuch recurring decimal fractions, particularly in multiplication and diviſion, the work will either be longer than neceſſary, or be very inaccurate, if the numbers are not conſidered as circulating ones: and to come at the true reſults of ſuch operations, ſeveral authors have given preciſe rules; and ſome of them have ſhewn the principles upon which thoſe rules were founded.

In the annexed paper thoſe principles are, endeavoured to be, exhibited in a different, and in a more general and conciſe manner, than has hitherto been ſhewn: but the modes of working are not here annexed, as they are to be found in *Cunn*, *Malcolm*, *Marſh*, and others; and may hereafter be fully exemplified in a treatiſe of Arithmetic, by the author hereof, conſidered in a more mathematical order, than what has hitherto been appropriated to this moſt uſeful ſcience.

General Principles.

1. Number is ſuppoſed to begin at unity, and from thence to aſcend and deſcend: thoſe terms aſcending above unity, are integers; and thoſe deſcending below unity, are fractions.

When

When in the aſcending and deſcending parts of the ſcale, the gradation proceeds by a tenfold value from the right hand towards the left, the rank of numbers, thus generated, is called the decimal ſcale.

As every place in this ſcale is ten times the value of its next right hand place; therefore the firſt place in thefractional part, is $\frac{1}{10}$ of the place of units; and the ſecond, third, fourth, &c. deſcending places in the fractional part, is $\frac{1}{100}$, $\frac{1}{1000}$, $\frac{1}{10000}$, &c. part of the place of units.

Therefore every decimal fraction is equal to a ſeries ariſing from m ultiplying the firſt, ſecond, third, fourth, &c. terms of the decreaſing geometrical progreſſion $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000}$, &c. by the firſt, ſecond, third, and fourth, &c. terms in the given fraction reſpectively.

Thus. Let the given fraction be 0,3587 or $\frac{3587}{10000}$.

$$\text{Then } 0{,}3587 = \frac{1}{10} \times 3 + \frac{1}{100} \times 5 + \frac{1}{1000} \times 8 + \frac{1}{10000} \times 7$$
$$= \frac{3}{10} + \frac{5}{100} + \frac{8}{1000} + \frac{7}{10000}$$
$$= \frac{3000}{10000} + \frac{500}{10000} + \frac{80}{10000} + \frac{7}{10000} = \frac{3587}{10000}.$$

2. Every decimal fraction ariſes from diviſion, when the dividend is leſs than the diviſor.

For, diviſor : dividend :: 10 : firſt term of the fraction;
:: 100 : ſum of the firſt and ſecond terms;
:: 1000 : ſum of the firſt, ſecond, and third, terms, &c.

And according to the ratio of the diviſor to the dividend, the quotient, or decimal fraction, will be finite or infinite.

3. Among thoſe decimal fractions which are infinite, or do not end, ſome of them *recur*, or circulate; that is, the ſame figure or figures run over again and again *ad infinitum*.

As 0,333 &c.; 0,2323 &c.; 0,758758 &c.; 0,999 &c.

Here 0,333 &c. $= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000}$, &c.

0,2323 &c. $= \frac{2}{10} + \frac{3}{100} + \frac{2}{1000} + \frac{3}{10000}$, &c.

0,785785 &c. $= \frac{7}{10} + \frac{8}{100} + \frac{5}{1000} + \frac{7}{10000} + \frac{8}{100000} + \frac{5}{1000000}$, &c.

0,999, &c. $= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000}$, &c.

The circulating fraction 0,999, &c. is equal to 1,0.

For the difference between 1,0 and 0,999 &c. is less than can be assigned.

To save a repetition of figures, it is usual to mark the first and last of circulating expressions, with points over the figures.

Thus, 0,333 &c. is wrote $0,\dot{3}$
0,2323 &c. $0,\dot{2}\dot{3}$
0,785785 &c. $0,\dot{7}8\dot{5}$

4. Like circulating decimal fractions are those which have each the same number of circulating places; and begin to recur each at the same name.

Every finite decimal fraction may be considered as infinite; cyphers being used as the circulating part.

Either place of a circulating expression may be taken as the first; observing that the number and order of the circulating places be not altered.

For as the decimal fraction arises by division; if either place of the recurring figures be taken for the first, the others will from thence regularly circulate.

Hence several unlike circulating decimal fractions may be made to begin and end at places of like names.

5. If in the decimal scale 10, 100, 1000, 10000, 100000, 1000000, &c. continued indefinitely, be selected any rank of equi-distant terms, such, that whatever term therein is taken for the first term, and the first term is made the common ratio to the rest; then will the sum of the reciprocals of those terms, be equal to the reciprocal of the number which is unity less than the first term.

Thus $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \&c. = \frac{1}{9}$; 10 being the 1st term.
$\frac{1}{100} + \frac{1}{10000} + \frac{1}{1000000} + \&c. = \frac{1}{99}$; 100 being the 1st term.
$\frac{1}{1000} + \frac{1}{1000000} + \frac{1}{1000000000} + \&c. = \frac{1}{999}$; 1000 being the 1st term.

For

For $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \&c. = \frac{100 + 10 + 1}{1000} = \frac{111\ \&c.}{1000\ \&c.}$

$\frac{1}{100} + \frac{1}{10000} + \frac{1}{1000000} + \&c. = \frac{10000 + 100 + 1}{1000000} = \frac{10101\ \&c.}{1000000\ \&c.}$

$\frac{1}{1000} + \frac{1}{1000000} + \frac{1}{1000000000} + \&c. = \frac{1000000 + 1000 + 1}{1000000000} = \frac{1001001}{1000000000\ \&c.}$

But $111\ \&c. \times 9 = 1000\ \&c.$ (by 3[d]) Then $\frac{111\ \&c.}{1000\ \&c.} = \frac{1}{9}$

$10101\ \&c. \times 99 = 100000\ \&c.$ Then $\frac{10101\ \&c.}{1000000\ \&c.} = \frac{1}{99}$

$1001001\ \&c. \times 999 = 1000000000\ \&c.$ Then $\frac{1001001\ \&c.}{1000000000\ \&c.} = \frac{1}{999}$

Hence the reciprocal of a number confifting of n places of 9's, is equal to a circulating number of n places, the right hand figure being 1, and the reft 0's.

Thus, $\frac{1}{9} = \frac{111\ \&c.}{1000\ \&c.} = 0,\dot{1}$

$\frac{1}{99} = \frac{10101\ \&c.}{1000000\ \&c.} = 0,\dot{0}\dot{1}$

$\frac{1}{999} = \frac{1001001\ \&c.}{1000000000\ \&c.} = 0,\dot{0}0\dot{1}$

$\frac{1}{9999} = \frac{10001001\ \&c.}{1000000000000\ \&c.} = 0,\dot{0}00\dot{1}$

6. If the reciprocal of a number confifting of n places of 9's, be multiplied by a number D, not exceeding n places; the product will be a circulating decimal fraction of n places, the right hand ones being the fame digits as are in the number D.

Let D = 3; or D = 23; or D = 785; or to any other number.

Now $\frac{1}{9} = \frac{111\ \&c.}{1000\ \&c.}$; $\frac{1}{99} = \frac{10101\ \&c.}{1000000\ \&c.}$; $\frac{1}{999} = \frac{1001001\ \&c.}{1000000000\ \&c.}$ (by 5th).

Therefore $\frac{1}{9}$, or $\frac{111\ \&c.}{1000\ \&c.} \times 3 = \frac{333\ \&c.}{1000\ \&c.} = 0{,}\dot{3}$

$$\frac{1}{99},\ \text{or}\ \frac{10101\ \&c.}{1000000\ \&c.}\left\{\begin{array}{lll}\times\ 3 & = \frac{30303\ \&c.}{1000000\ \&c.} & = 0{,}\dot{0}\dot{3} \\ \times\ 23 & = \frac{232323\ \&c.}{1000000\ \&c.} & = 0{,}\dot{2}\dot{3}\end{array}\right.$$

$$\frac{1}{999},\ \text{or}\ \frac{1001001\ \&c.}{1000000000\ \&c.}\left\{\begin{array}{lll}\times\ 3 & = \frac{3003003\ \&c.}{1000000000\ \&c.} & = 0{,}\dot{0}0\dot{3} \\ \times\ 23 & = \frac{23023023\ \&c.}{1000000000\ \&c.} & = 0{,}\dot{0}2\dot{3} \\ \times\ 785 & = \frac{785785785\ \&c.}{1000000000\ \&c.} & = 0{,}\dot{7}8\dot{5}\end{array}\right.$$

7. Hence every circulating decimal fraction will be equivalent to a vulgar fraction, wherein the numerator is thoſe circulating figures, and the denominator conſiſts of as many 9's, as are figures in the numerator.

Thus $0{,}\dot{3} = \frac{3}{9}$. For $\frac{1}{9} \times 3 = \frac{3}{9} = 0{,}\dot{3}$ (by 6th)

$0{,}\dot{0}\dot{3} = \frac{3}{99}$ $\quad \frac{1}{99} \times 3 = \frac{3}{99} = 0{,}\dot{0}\dot{3}$

$0{,}\dot{2}\dot{3} = \frac{23}{99}$ $\quad \frac{1}{99} \times 23 = \frac{23}{99} = 0{,}\dot{2}\dot{3}$

8. Hence a circulating decimal fraction, of any number of places, being multiplied by a number of as many 9's, will give a finite expreſſion, having the ſame figures as are in the circulating one.

Thus $0{,}\dot{6} \times 9 = 6$. For $9 : 1 :: 6 : 0{,}\dot{6}$

$0{,}\dot{0}\dot{6} \times 99 = 6$. $\quad 99 : 1 :: 6 : 0{,}\dot{0}\dot{6}$

$0{,}\dot{2}\dot{5} \times 99 = 25$. $\quad 99 : 1 :: 25 : 0{,}\dot{2}\dot{5}$

$0{,}\dot{6}2\dot{5} \times 999 = 625$. $\quad 999 : 1 :: 625 : 0{,}\dot{6}2\dot{5}$

Hence

Hence it appears, that, in common multiplication, the product of a circulating number, by its proper denominator, in 9's, will be deficient of the true product by that circulating number.

Thus $0,\dot{6} \times 9 = 5,4$; then $5,4 + ,6 = 6$. For $\frac{6}{9} = 0,\dot{6}$

$0,0\dot{6} \times 99 = 5,94$; then $5,94 + 0,06 = 6$. For $\frac{6}{99} = 0,0\dot{6}$

$0,6\dot{2}\dot{5} \times 999 = 624,375$; then $624,375 + 0,625 = 625,0$

Hence. Any finite number is in proportion to the same number recurring, as the proper denominator of the circulate is to that denominator increased by unity.

Thus $9 : 10 :: 6 : \dot{6}$. For $\dot{6} \times 9 = 6 \times 10$

$99 : 100 :: 25 : \dot{2}\dot{5}$. For $\dot{2}\dot{5} \times 99 = 25 \times 100$.

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If to the preceding articles, be joined the compendiums of multiplying and dividing by any number of 9's, they will constitute the whole of the theory, upon which depend all the operations with circulating numbers: for as these have 9's for their denominator, wanting unity in the lowest place to make them 10's; therefore unity for every 9 is applied in some additions and multiplications: Or, the circulating parts being reduced to finite number; then working with them by the common rules, will give finite results; which results are to be reduced to circulates by contrary operations to what were used to reduce the circulates to finites.

XXXIII. *A*